UDC 531.31

# SOLUTION OF ONE OF BIRKHOFF'S PROBLEMS

PMM Vol. 42, No. 6, 1978, pp. 1136-1141
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(Received December 15, 1977)

The criterion of existence of an integral linear with respect to generalized velocities over the set of trajectories with the same total energy is proved for conservative dynamic systems with two degrees of freedom.

1. Birkhoff had formulated the problem of determining the conditions under which the Lagrange equations [1, 2] composed for function

$$L = T_2 + T_1 + T_0 + U, \quad T_2 = {}^1/_2 a_{ij} q^{\cdot i} q^{\cdot j}, \quad T_1 = a_i q^{\cdot i} \quad (q^\cdot = dq/dt) \tag{1.1}$$

($a_{ij}$, $a_i$, $T_0$, and $U$ are functions of generalized coordinates $q^1$ and $q^2$ of the dynamic system, and the recurrent indices denote summation in the range from unity to two) together with the supplementary equation

$$T_2 - T_0 - U = h$$

have an integral of the form

$$\xi_i(q^1, q^2)\, q^{\cdot i} + \xi(q^1, q^2) = \text{const} \tag{1.2}$$

which in [2] was called conditional. It generally exists only for certain fixed values of the constant $h$.

In a two-dimensional Riemennian space with linear element

$$ds^2 = 2T_2 dt^2 = a_{ij} dq^i dq^j$$

it is always possible to introduce isothermal coordinates $R^1$ and $R^2$ [3] in which

$$ds^2 = \rho(R^1, R^2)\, [(dR^1)^2 + (dR^2)^2]$$

Because of this Birkhoff assumed that the Lagrangian (1.1) is already specified in isothermal coordinates. It was shown on this basis in [1, 2] that integral (1.2) makes possible a point transformation of coordinates and a nonholonomic substitution of the independent variable integral, such that in the new variables $Q^1$ and $Q^2$, $\tau$ the Lagrangian $L + q^{\cdot i}\partial_i b$ ($\partial_i = \partial / \partial q^i$, $b(q^1, q^2)$ is some function) assumes the form

$$L_B = \frac{1}{2}\left[\left(\frac{dQ^1}{d\tau}\right)^2 + \left(\frac{dQ^2}{d\tau}\right)^2\right] + \alpha\, \frac{dQ^1}{d\tau} + \beta\, \frac{dQ^2}{d\tau} + \gamma \tag{1.3}$$

with the ignorable coordinate. The converse statement is evidently also true. The point transformation of coordinates and the substitution of $dt$ in [1, 2] are expressed in terms of coefficients of the sought linear integral, hence we only have an equivalent formulation of the problem posed above.

Isothermal coordinates are not uniquely determined [3], since the variables $\operatorname{Re}[f(R^1 + iR^2)]$ and $\operatorname{Im}[f(R^1 + iR^2)]$, where $f$ is an arbitrary analytic function ($i = \sqrt{-1}$), are also isothermal coordinates. The formulas for passing from the specified generalized coordinates $q^1$ and $q^2$ to a system of some isothermal coordinates can, for instance, be achieved by integrating the differential equation

$$a_{11}\,dq^1 + \left(a_{12} + \sqrt{(a_{12})^2 - a_{11}a_{12}}\right) dq^2 = 0$$

and take the real and imaginary parts of its general integral as the new coordinates. Other known methods of passing to isothermal coordinates are also associated with integration of differential equations. Because of this it is interesting to establish the criterion of existence of variables

$$Q^1 = Q^1 (q^1, q^2), \quad Q^2 = Q^2 (q^1, q^2) \tag{1.4}$$

$$d\tau = \frac{dt}{\lambda (q^1, q^2)} \tag{1.5}$$

in which the Lagrangian is of the form (1.3) with an ignorable coordinate, and $q^1$ and $q^2$ are arbitrarily selected generalized coordinates of the system. The problem of Birkhoff is solved below in such formulation.

Note that transformation (1.4) preserves the Lagrangian form of equations, while the substitution (1.5) generally does not, as shown by that the Lagrange equations constituted for function (1.1) assume, after the substitution (1.5), the form

$$\frac{d}{d\tau}\frac{\partial T^*}{\partial q^{j'}} - \frac{\partial T^*}{\partial q^j} - \frac{\partial}{\partial q^j}(\lambda U) - \left(\frac{T_2^*}{\lambda} - T_0 - U\right)\frac{\partial \lambda}{\partial q^j} = 0 \quad (j = 1, 2) \tag{1.6}$$

$$q^{j'} = \frac{dq^j}{d\tau}, \quad T^* = T_2^* + T_1^* + \lambda T_0, \quad T_2^* = \frac{1}{2\lambda} a_{ij} q^{i'} q^{j'}, \quad T_1^* = a_i q^{i'}$$

However, by restricting the analysis to isoenergetic trajectories of the system [1, 2, 4] Eqs. (1.6) can be reduced to the form of Lagrange equations in terms of functions

$$L^* = T_2^* + T_1^* + \lambda (T_0 + U + h) \tag{1.7}$$

Comparison of terms quadratic with respect to velocities in formulas (1.3) and (1.7) of the system Lagrangian shows that coordinates (1.4) are isothermal.

2. Let us consider the case of the reversible dynamic system [1, 2], i.e. when $T = T_2$.

We introduce the Riemannian space metric

$$dl^2 = (U + h)\, a_{ij} dq^i dq^j = b_{ij} dq^i dq^j \tag{2.1}$$

which corresponds to the principle of least action in Jacobi's form. The Gaussian curvature of that space is [5]

$$K = \frac{1}{\delta^4}\begin{vmatrix} b_{11}, & \partial_1 b_{11}, & \partial_2 b_{11} \\ b_{12}, & \partial_1 b_{12}, & \partial_2 b_{12} \\ b_{22}, & \partial_1 b_{22}, & \partial_2 b_{22}\end{vmatrix} - \frac{1}{2\delta}\left[\partial_2 \left(\frac{\partial_2 b_{11} - \partial_1 b_{12}}{\delta}\right) - \partial_1 \left(\frac{\partial_2 b_{12} - \partial_1 b_{22}}{\delta}\right)\right] \tag{2.2}$$

$(\delta^2 = b_{11} b_{22} - (b_{12})^2)$

We denote $|\operatorname{grad} K|^2$ and $\operatorname{div}\operatorname{grad} K$ respectively by

$$\Delta_1 K = b^{ij}\,\partial_i K \partial_j K, \quad \Delta_2 K = \frac{1}{\delta}\,\partial_i (\delta b^{ij} \partial_j K) \tag{2.3}$$

The following theorem is valid.

Theorem 1. For the existence in the reversible dynamic system of integral (1.2), conditional in Birkhoff's meaning, it is necessary and sufficient if for $h$ fixed with respect to $q^1$ and $q^2$ the following conditions are satisfied:

$$K \equiv \text{const} \tag{2.4}$$

$$J(\Delta_1 K, K) = 0, \quad J(\Delta_2 K, K) = 0 \tag{2.5}$$

($J(f, w)$ is the Jacobian of functions $f$ and $w$).

P r o o f. T h e n e c e s s i t y. If integral (1.2) obtains for some value of the constant $h$, the Lagrange function of the dynamic system in coordinates (1.4) is of the form

$$L = \varkappa(Q^1, Q^2)[(Q^{\cdot 1})^2 + (Q^{\cdot 2})^2] + U(Q^1, Q^2)$$

After effecting the substitution (1.5) with $\lambda = \varkappa$ we obtain a Lagrangian of the form

$$L^* = \frac{1}{2}\left[\left(\frac{dQ^1}{d\tau}\right)^2 + \left(\frac{dQ^2}{d\tau}\right)^2\right] + \varkappa(U + h) \tag{2.6}$$

in which function $\varkappa(U + h)$ does not contain at least one of coordinates $Q^1$ or $Q^2$. Consequently, (2.1) represents the metric of rotation [3, 6], since in coordinates $Q^1$ and $Q^2$

$$dl^2 = \varkappa(U + h)[(dQ^1)^2 + (dQ^2)^2] \tag{2.7}$$

Formulas (2.4) and (2.5) represent the known necessary and sufficient conditions for an arbitrary metric of the form (2.1) to be a metric of rotation.

T h e s u f f i c i e n c y. Let us assume that condition (2.4) or (2.5) is satisfied for some $h$. Then (2.1) represent the metric of rotation and there exist generalized coordinates (1.4) in which it is expressed in the form (2.7) with an ignorable coordinate, for example, $Q^2$. In the variables $Q^1, Q^2$, and $\tau$ ($dt = \varkappa d\tau$) the Lagrangian of the system assumes the form (2.2). Hence the relation $Q^{2\prime} = \text{const}$ is the integral of the system in the set of iso-energetic trajectories that correspond to the considered value of $h$. The theorem is proved.

R e m a r k s. 1°. The checking of condition (2.5) can be simplified by substituting some function $\Phi(K)$ for curvature $K$. The result remains unchanged, since for any arbitrary function $\Phi(K)$

$$\Delta_1\Phi(K) = [\Phi'(K)]^2\Delta_1 K, \quad \Delta_2\Phi(K) = \Phi''(K)\,\Delta_1 K + \Phi'(K)\,\Delta_2 K \tag{2.8}$$

2°. Let us assume that $K \not\equiv \text{const}$ and that conditions (2.5) of the theorem are satisfied. Using appropriate geometric data [3, 6] it is not difficult to establish that the most general point transformation that reduces the linear element (2.1) to the form of ignorable coordinate $Q^2$ is provided by formulas

$$Q^1 = \eta(\Phi), \quad Q^2 = \int \mu\,[(b_{21}\partial_1\Phi - b_{11}\partial_2\Phi)\,dq^1 + (b_{22}\partial_1\Phi - b_{12}\partial_2\Phi)\,dq^2] + \psi(\Phi), \quad \mu = \frac{1}{\delta}\exp\left(-\int \frac{\Delta_2\Phi}{\Delta_1\Phi}\,d\Phi\right) \tag{2.9}$$

where $\Phi(K)$, $\eta$, and $\psi$ are arbitrary independent functions. Obviously the coordinates $Q^1$ and $Q^2$ are isothermal then and only then when

$$\psi \equiv 0, \quad \eta(\Phi) = \pm\int \exp\left(-\int \frac{\Delta_2\Phi}{\Delta_1\Phi}\,d\Phi\right)d\Phi \tag{2.10}$$

If $K \equiv \text{const}$, then instead of function $\Phi(K)$ which generates transformation (2.9) it is possible to take any nontrivial solution $y(q^1, q^2)$ of the system of equations

$$dy_i = (\Gamma_{ri}^{s} y_s - K y b_{ir})\, dq^r, \quad dy = y_i dq^i \tag{2.11}$$

$$\Gamma_{ri}^{s} = {}^1/_2\, b^{sj} (\partial_i b_{rj} + \partial_r b_{ij} - \partial_j b_{ir})$$

which for $K \equiv \text{const}$ represents a system in total differentials with three unknown functions $y$, $y_1$, and $y_2$ [6].

3°. According to Levi's theorem [7] extended by Whittaker [8] conditions (2.4) and (2.5) provide a criterion of existence of the linear integral of equations that are geodesic in the space with metric (2.1). When $K \not\equiv \text{const}$, this Birkhoff's conditional integral is determined by a single quadrature in variables $q^1, q^2$, and $t$ is of the form

$$\frac{b^{ij} v_j \partial_i K}{\Delta_1 K} \exp\left(\int \frac{\Delta_2 K}{\Delta_1 K}\, dK\right) = \text{const} \quad (v_1 = -q^{\cdot 2}\delta,\ v_2 = q^{\cdot 1}\delta)$$

This integral is general, since its total time derivative vanishes by virtue of Lagrange equations, generally not identically, but only on isoenergetic trajectories of the system that correspond to some specific values of constant $h$. Its constant of integration is, however, arbitrary.

Example. Let

$$T = \frac{1}{2\varphi}(x^{\cdot 2} + y^{\cdot 2}), \quad U = \varphi$$

where function $\varphi(x, y)$ is positive and does not satisfy any equation of the form

$$(\partial\varphi / \partial x)^2 + (\partial\varphi / \partial y)^2 = F(\varphi)$$

The system has no general linear integrals, since the necessary condition $J(\Delta_1 U, U) = 0$ [9] is not satisfied. However, if $h = 0$, there are three conditional linear integrals

$$x^{\cdot} / \varphi = \text{const}, \quad y^{\cdot} / \varphi = \text{const}, \quad 1 / \varphi\,(x^{\cdot} y - y^{\cdot} x) = \text{const}$$

3. Let us consider the general case of (1.1). We replace (2.1) by the metric

$$dl^2 = (T_0 + U + h)\, a_{ij} dq^i dq^j = b_{ij} dq^i dq^j \tag{3.1}$$

and denote by $K$ the Gaussian curvature of the two dimensional space with that metric, and by a the vector with covariant components $a_1$ and $a_2$.

Theorem 2. Let $K \not\equiv \text{const}$. The Birkhoff's conditional integral (1.2) exists then and only then when conditions (2.5) are satisfied and

$$J(\text{rot a}, K) = 0, \quad \text{rot a} = \frac{1}{\delta}(\partial_1 a_2 - \partial_2 a_1) \tag{3.2}$$

Proof. Conditions (2.5) are necessary (the proof is similar to that of Theorem 1, but for the case of metric (3.1)). If the Lagrangian $L + q^{\cdot i}\partial_i b$ in variables $Q^1$, $Q^2$, and $\tau$ is of the form (1.3) in which, for instance, $Q^2$ is the ignorable coordinate, then, as already noted, coordinates $Q^1$ and $Q^2$ must be related to the input coordinates $q^1$ and $q^2$ by formulas (2.9) and (2.10). Hence, taking into account the expressions for $T_1^*$, it is necessary that

$$(a_1 + \partial_1 b)\, dq^1 + (a_2 + \partial_2 b)\, dq^2 = \chi\mu\, [(b_{21}\partial_1\Phi - b_{11}\partial_2\Phi)\, dq^1 +$$
$$(b_{22}\partial_1\Phi - b_{12}\partial_2\Phi)\, dq^2]$$

where $\chi$ is some function of $\Phi$. From this we find that

$$\operatorname{rot} \mathbf{a} = \Delta_1\Phi \frac{d}{d\Phi}\left[\chi \exp\left(-\int \frac{\Delta_2\Phi}{\Delta_1\Phi}\, d\Phi\right)\right] + \Delta_2\Phi\chi \exp\left(-\int \frac{\Delta_2\Phi}{\Delta_1\Phi}\, d\Phi\right)$$

i.e. $\operatorname{rot} \mathbf{a}$ is a function of $K$.

Converse reasoning is used for proving the sufficiency conditions (2.5) and (3.2). The theorem is proved.

Remarks. 1°. If $K \equiv \text{const}$, then the necessary and sufficient condition of existence of Birkhoff's conditional integral (1.2) is the existence of the nontrivial solution $y(q^1, q^2)$ of Eqs. (2.11) for which

$$J(\operatorname{rot} \mathbf{a}, y) = 0 \tag{3.3}$$

Test of this condition can be carried out without prior integration of Eqs. (2.11).

2°. Birkhoff had noted [1] that when the system has a conditional linear integral and $\operatorname{rot} \mathbf{a} \not\equiv \text{const}$, the set of lines $\operatorname{rot} \mathbf{a} = \text{const}$ belongs to some isothermal net on the characteristic surface. The coordinate lines $Q^1 = \text{const}$ and $Q^2 = \text{const}$ can be selected as the coordinate lines of that net, so that an ignorable coordinate would appear in the related Lagrange function (1.3).

As previously shown, such isothermal net is the sought parametrization defined by Eqs. (2.9) and (2.10) that are also valid in the case of $\operatorname{rot} \mathbf{a} \equiv \text{const}$.

4. Let $K \not\equiv \text{const}$. According to (3.1) and (2.2) $K$ is a rational expression of $h$ whose numerator is a second power polynomial of $h$ and the denominator is the product of $(U + h)^3$ into some function of coordinates which does not contain $h$. Consequently formulas (2.5) and (3.2) yield algebraic equations of the tenth, eighth, and fourth powers of $h$ with coefficients that are functions of coordinates. This proves the following theorem.

Theorem 3. When $K \not\equiv \text{const}$, the number of different values of the constant of the integral for which conditional linear integrals exist cannot be higher than ten. This is evidently a very rough estimate.

If conditions (2.5) and (3.2) or (2.4) and (3.3) are identically satisfied with respect to $h$, the system Lagrangian can be reduced to the form with an ignorable coordinate only by point transformation. The corresponding necessary and sufficient conditions for the force function in the case of reversible systems with two degrees of freedom were obtained in [10], and in terms of operators $\Delta_1$ and $\Delta_2$ (2.3) are given in [9].

5. Having chosen some isothermal coordinates (1.4) it is possible, using (1.5), to reduce the system Lagrangian to the form (1.3) in which $\gamma = \lambda(T_0 + U + h)$. Function (1.3) then generally depends on both coordinates $Q^1$ and $Q^2$. Hence it is possible to formulate the problem of determination of conditions of existence in the dynamic system of the conditional linear integral for the Lagrangian (1.3) without loss of generality. This approach was used in [1, 2, 11, 12]. Two necessary conditions of the considered problem were obtained in [11, 12] in a form that permits their test. They are of the same form for $s = \ln \gamma$ and $s = \ln \omega$

$$\left(s_\xi + 2\frac{\sigma_{\eta\eta}\sigma_\xi - \sigma_{\xi\eta}\sigma_\eta}{\sigma_\xi^2 + \sigma_\eta^2}\right)\sigma_\eta = \left(s_\eta + 2\frac{\sigma_{\xi\xi}\sigma_\eta - \sigma_{\xi\eta}\sigma_\xi}{\sigma_\xi^2 + \sigma_\eta^2}\right)\sigma_\xi \tag{5.1}$$

$$\xi = Q^1, \quad \eta = Q^2, \quad \sigma = (\Delta_{\xi\xi} + s_{\eta\eta}) \exp(-\Delta), \quad \omega = \alpha_\eta - \beta_\xi$$

(the subscripts denote variables with respect to which partial differentiation is carried out).

It can be shown, however, that these conditions are not sufficient, since in coordinates $\xi$, and $\eta$ the linear element (3.1) $dl^2 = \gamma (d\xi^2 + d\eta^2)$ , hence for $\Delta = \ln \gamma$ we have $\sigma = -2K$, $(\sigma_\xi{}^2 + \sigma_\eta{}^2) \exp(-s) = 4\Delta_1 K$ and condition (5.1) of the form

$$J(\Delta_1 K, K) = 0 \tag{5.2}$$

For $s = \ln \omega$ we similarly have $\sigma = [-2K + \Delta_2 (\ln \text{rot a})] / \text{rot a}$, $\omega = \gamma$ rot a , and condition (5.1) of the form

$$J\left(\frac{\Delta_1\sigma}{\text{rot a}}, \sigma\right) = 0 \tag{5.3}$$

In conformity with (2.5), (3.2) and (2.8) conditions (5.2) and (5.3) must actually be satisfied, but they do not contain the second necessary condition from (2.5) which is independent of the remaining conditions (2.5) and (3.2). Because of this, by Theorem 2, these conditions are not sufficient.

The author thanks V. V. Rumiantsev for useful remarks and discussion of results.

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Translated by J.J. D.